Homework 5 Solution

1. Sec. 2.2 Q15

15. Let V and W be vector spaces, and let S be a subset of V. Define $S^0 = \{\mathsf{T} \in \mathcal{L}(\mathsf{V}, \mathsf{W}) \colon \mathsf{T}(x) = 0 \text{ for all } x \in S\}$. Prove the following statements.

(a) S^0 is a subspace of $\mathcal{L}(\mathsf{V}, \mathsf{W})$.
(b) If $S_1$and S_2 are subsets of V and $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.
(c) If V_1 and V_2 are subspaces of V, then $(\mathsf{V}_1 + \mathsf{V}_2)^0 = \mathsf{V}_1^0 \cap \mathsf{V}_2^0$.

(a) • Let T_0 be the zero transformation from V to W.
Then $T_0(x) = 0 \quad \forall x \in S \subset V$.
So $T_0 \in S^0$

• $\forall T_1, T_2 \in S^0. \quad \forall c \in F$
$(cT_1 + T_2)(x) = c \cdot T_1(x) + T_2(x) = c \cdot 0 + 0 = 0 \quad \forall x \in S$
So $cT_1 + T_2 \in S^0$

Thus S^0 is a subspace of $\mathcal{L}(V, W)$

(b) $\forall T \in S_2^0. \quad T(x) = 0 \quad \forall x \in S_2$
Since $S_1 \subset S_2$, $\quad T(x) = 0 \quad \forall x \in S_1 \subset S_2$
Thus $T \in S_1^0$. i.e. $S_2^0 \subset S_1^0$

(c) • Since $V_1 \subset V_1 + V_2$, $V_2 \subset V_1 + V_2$
By (b) we have $(V_1+V_2)^0 \subset V_1^0$ and $(V_1+V_2)^0 \subset V_2^0$
Then $(V_1+V_2)^0 \subset V_1^0 \cap V_2^0$

• $\forall T \in V_1^0 \cap V_2^0$ Then $T(x) = 0 \quad \forall x \in V_1 \cup V_2$
Then for any $y \in V_1 + V_2$, there exist $x_1 \in V_1$, $x_2 \in V_2$
such that $y = x_1 + x_2$
Then $T(y) = T(x_1) + T(x_2) = 0 + 0 = 0$
Thus $T \in (V_1+V_2)^0$ i.e. $V_1^0 \cap V_2^0 \subset (V_1+V_2)^0$

2. Consider a linear transformation $T : V \longrightarrow W$. Prove or disprove the following.

(a) If T has a right inverse, must it have a left inverse?

(b) If T has a left inverse, must it have a right inverse?

(c) If T has both a left and a right inverse, must it be invertible? (That is, must the left and right inverse be the same?)

(d) If T has a unique right inverse S, is T necessarily invertible? (Hint. Consider $ST + S - I$.)

(a) No.

$$T: F^\infty \longrightarrow F^\infty$$
$$(a_1, a_2, \dots) \longmapsto (a_2, a_3, \dots)$$
$$U: F^\infty \longrightarrow F^\infty$$
$$(a_1, a_2, \dots) \longmapsto (0, a_1, a_2, \dots)$$

Then $T \circ U = I$, T has a right inverse.

Suppose T has left inverse F then $F \circ T = I$.

$$\begin{cases} (U \circ T) \circ F((a_1, a_2, \dots)) = I \circ F((a_1, a_2, \dots)) \\ \qquad = F((a_1, a_2, \dots)) \\ U \circ (T \circ F)((a_1, a_2, \dots)) = U \circ I((a_1, a_2, \dots)) \\ \qquad = U((a_1, a_2, \dots)) \end{cases}$$

Thus $U = F$

but $U \circ T((1, 0, 0, \dots)) = U((0, 0, \dots)) = (0, 0, \dots)$

i.e. $U \circ T = I$

contradiction. so T has no left inverse.

(b) No.

Consider
$$T: F^\infty \rightarrow F^\infty$$
$$(a_1, a_2, \dots) \longmapsto (0, a_1, a_2, \dots)$$
$$U: F^\infty \rightarrow F^\infty$$
$$(a_1, a_2, \dots) \longmapsto (a_2, a_3, \dots)$$

Then $U \circ T = I$ i.e. T has left inverse U.

By similar argument in (a), we know that T does not have a right inverse.

(c) if T has a left inverse U and a right inverse S

Then $U = U \circ I = U \circ (T \circ S) = (U \circ T) \circ S = I \circ S = S$

(d) T has a unique S. Then $T \circ S = I$.

$$T \circ (S \circ T + S - I) = T \circ S \circ T + T \circ S - T$$
$$= I \circ T + T \circ S - T$$
$$= I$$

Hence $S = S \circ T + S - I$

Then $S \circ T = I$, S is also the left inverse of T

T is invertible.

3. Consider a linear transformation $T : V \longrightarrow W$, where $dim(V) = dim(W) = n$. Show that if T has a left inverse U, then U is also a right inverse of T, thus T is invertible. (Hint. Sec. 2.4 Q10(b), prove it if you use it)

$T : V \to W$ has left inverse $U : W \to V$

Then $U \circ T = I_V$

$\forall v \in N(T)$, $T(v) = 0$.

Then $v = I_V(v) = U \circ T(v) = U(0) = 0$.

i.e. T is 1-1

Since $\dim(V) = \dim(W)$. we have T is invertible.

$\exists\, T^{-1} : W \to V$. s.t $T \circ T^{-1} = I_W$ $T^{-1} \circ T = I_V$

$$T^{-1} = I_V \circ T^{-1} = (U \circ T) \circ T^{-1} = U \circ I_W = U$$

So U is also a right inverse of T.

4. Sec. 2.4 Q20

20.† Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let β and γ be ordered bases for V and W, respectively. Prove that $\operatorname{rank}(\mathsf{T}) = \operatorname{rank}(\mathsf{L}_A)$ and that $\operatorname{nullity}(\mathsf{T}) = \operatorname{nullity}(\mathsf{L}_A)$, where $A = [\mathsf{T}]_\beta^\gamma$. *Hint:* Apply Exercise 17 to Figure 2.2.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] \\ F^n \arrow[r, "L_A"] & F^m \end{tikzcd} \qquad \phi_\gamma \circ T = L_A \circ \phi_\beta$$

$\phi_\beta : V \to F^n$, $v \mapsto [v]_\beta$ and $\phi_\gamma : W \to F^m$, $w \mapsto [w]_\gamma$ are isomorphism.

$$\operatorname{rank}(T) = \dim(R(T)) = \dim(T(V))$$

by ex 17

$$\begin{aligned} &= \dim(\phi_\gamma(T(V))) \\ &= \dim(L_A \circ \phi_\beta(V)) \\ &= \dim(L_A(F^n)) \\ &= \dim(R(L_A)) \\ &= \operatorname{rank}(L_A) \end{aligned}$$

$$\begin{aligned} \operatorname{nullity}(T) &= \dim(V) - \operatorname{rank}(T) \\ &= \dim(F^n) - \operatorname{rank}(L_A) \\ &= \operatorname{nullity}(L_A) \end{aligned}$$

5. Sec. 2.4 Q24

24. Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{Z}$ be a linear transformation of a vector space V onto a vector space Z. Define the mapping

$$\overline{\mathsf{T}}\colon \mathsf{V}/\mathsf{N}(\mathsf{T}) \to \mathsf{Z} \quad \text{by} \quad \overline{\mathsf{T}}(v + \mathsf{N}(\mathsf{T})) = \mathsf{T}(v)$$

for any coset $v + \mathsf{N}(\mathsf{T})$ in $\mathsf{V}/\mathsf{N}(\mathsf{T})$.

(a) Prove that $\overline{\mathsf{T}}$ is well-defined; that is, prove that if $v + \mathsf{N}(\mathsf{T}) = v' + \mathsf{N}(\mathsf{T})$, then $\mathsf{T}(v) = \mathsf{T}(v')$.
(b) Prove that $\overline{\mathsf{T}}$ is linear.
(c) Prove that $\overline{\mathsf{T}}$ is an isomorphism.
(d) Prove that the diagram shown in Figure 2.3 commutes; that is, prove that $\mathsf{T} = \overline{\mathsf{T}}\eta$.

$$\begin{tikzcd}
\mathsf{V} \arrow[rr, dashed, "\mathsf{T}"] \arrow[dr, dashed, "\eta"'] & & \mathsf{Z} \\
& \mathsf{V}/\mathsf{N}(\mathsf{T}) \arrow[ur, dashed, "\overline{\mathsf{T}}"'] &
\end{tikzcd}$$

Figure 2.3

(a) if $v + N(T) = v' + N(T)$, then $v - v' \in N(T)$

$T(v) - T(v') = T(v - v') = 0$

$\bar{T}(v + N(T)) = T(v) = T(v') = \bar{T}(v' + N(T))$

(b) $\bar{T}\Big(a\big(v_1 + N(T)\big) + \big(v_2 + N(T)\big)\Big)$

$= \bar{T}\Big((av_1 + v_2) + N(T)\Big) = T(av_1 + v_2)$

$= a \cdot T(v_1) + T(v_2)$

$= a \cdot \bar{T}(v_1 + N(T)) + \bar{T}(v_2 + N(T))$

(c) $\forall z \in Z$. since T is onto.

$\exists v \in V$ s.t. $z = T(v)$

$\exists v + N(T) \in V/N(T)$ s.t.

$\bar{T}(v + N(T)) = T(v) = z$.

Thus $\bar{T}$ is onto.

$\forall v + N(T) \in N(\bar{T})$.

$0 = \bar{T}(v + N(T)) = T(v)$, $v \in N(T)$

Then $v + N(T) = N(T)$ is the zero in $V/N(T)$

Thus $\bar{T}$ is 1-1

Therefore $\bar{T}$ is an isomorphism.

(d)

$$\bar{T} \circ \eta(v) = \bar{T}(v + N(T)) = T(v) \quad \forall v \in V.$$